Problem Set II: due Monday, February 5

1) Kulsrud, 4.1

2) Kulsrud, 4.2

3) Kulsrud, 4.3

4) Kulsrud, 4.5

5) Kulsrud, 5.1. Derive an Energy and Momentum Theorem for acoustic waves.

6) Kulsrud, 5.2. Ignore last sentence.

7) Kulsrud, 5.4

8) Kulsrud, 5.5

9) a) Show that for incompressible MHD in two dimensions, the basic equations can be written as:

$$(\partial_t + \underline{v}\cdot\nabla)\nabla^2\phi = (B\cdot\nabla)\nabla^2 A + \nu\nabla^2\nabla^2\phi + \tilde{f}$$

$$(\partial_t + \underline{v}\cdot\nabla)A = \eta\nabla^2 A.$$

Here ν is viscosity, η is resistivity, $\underline{v} = \underline{\nabla}\phi\times\hat{z}$ and $\underline{B} = \underline{\nabla}A\times\hat{z}$. $\tilde{f}$ is a random force. Take $P = P(\rho)$.

b) Take $\underline{B} = B_0\hat{x}$ to be a weak in-plane magnetic field. Calculate the real frequency and damping for Alfven waves.

c) Using quasilinear theory, calculate the turbulent resistivity induced by a spectrum of Alfven waves in 2D MHD. For $\nu\to 0$, interpret your result in terms of the freezing-in-law. Why does viscosity enter your result for part (i)? Why does η enter? Contrast these.

d) Taking $\underline{B} = B_0\hat{x}$ and $\langle\tilde{V}_y\tilde{A}\rangle = -\eta_T\,\partial A_0/\partial y$ as a definition of turbulent resistivity η_T. Show that at stationarity

$$\eta_T = \eta\langle\tilde{B}^2\rangle/B_0^2,$$

assuming the system has periodic boundary conditions. Discuss your result and its implications. This is a famous result, referred to as the Zeldovich Theorem, after Ya.B. Zeldovich.

e) What happens if one pair of boundaries are open? (Hint: Consider flux thru surface.)

10) a) Derive the tensor virial theorem for a warm, self-gravitating fluid in an external gravitational potential $\phi_{ext}(\underline{x})$. In particular, how does $\phi_{ext}(\underline{x})$ change the virial balance?

b) Describe the structure of $\phi_{ext}(\underline{x})$, relative to the blob, which is required to confine the fluid.

11) a) Derive reduced MHD by two different methods. Explain the physics.

b) What linear waves does reduced MHD support? What happened to the others – i.e. how does the ordering eliminate them? (N.B. It may be useful to read Strauss, '76).

c) Recover 2D MHD from reduced MHD.

d) What are the conservation laws of reduced and 2D MHD?

e) Now, derive the reduced MHD equations when $\underline{B}_o = B_o \hat{z}$ and gravity is present, i.e. $\underline{g} = g\,\hat{x}$.